Modal team logics for modelling Free Choice inference

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Helsinki Logic Seminar

Overview

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Aloni: Bilateral state-based modal logic (*BSML*) accounts for FC

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Natural deduction axiomatizations

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A possible formalization:

$$(\dagger) \qquad \Diamond(\phi \vee \psi) \to \Diamond\phi$$

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A possible formalization:

(†) $\Diamond(\phi \vee \psi) \rightarrow \Diamond\phi$

Problem:

1. $\Diamond p$
2. $\Diamond(p \vee q)$ (1, classical modal logic)
3. $\Diamond q$ (2, †)

Bilateral State-based Modal Logic

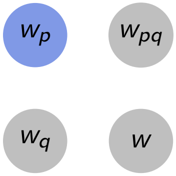
Team semantics for modal logic

$$M = (W, R, V)$$

standard Kripke semantics

$$M, w \vDash \phi$$

$$w \in W$$

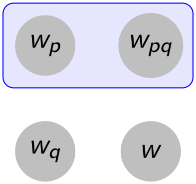


$$w_p \vDash p$$

state-based/team semantics

$$M, s \vDash \phi$$

$$s \subseteq W$$



$$\{w_p, w_{pq}\} \vDash p$$

Bilateralism

“ϕ is assertable in s”	$s \vDash \phi$
“ϕ is rejectable in s”	$s \dashv \phi$

Bilateralism

"ϕ is assertable in s"	$s \models \phi$
"ϕ is rejectable in s"	$s \dashv \phi$

Bilateral negation

$$s \models \neg\phi \iff s \dashv \phi$$

$$s \dashv \neg\phi \iff s \models \phi$$

Syntax of *BSML*:

$$\phi ::= p \mid \neg\phi \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid \Diamond \phi \mid \mathrm{NE}$$

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Semantics ($\vDash$)

$$\begin{array}{lcl} s \vDash p & \iff & \forall w \in s : w \in V(p) \\ s \vDash \neg\phi & \iff & s \dashv \phi \\ s \vDash \phi \wedge \psi & \iff & s \vDash \phi \text{ and } s \vDash \psi \\ s \vDash \phi \vee \psi & \iff & \exists t, t' : t \cup t' = s \text{ and } t \vDash \phi \text{ and } t' \vDash \psi \\ s \vDash \Diamond\phi & \iff & \forall w \in s : \exists t \subseteq R[w] : t \neq \varnothing \text{ and } t \vDash \phi \\ s \vDash \mathrm{NE} & \iff & s \neq \varnothing \end{array}$$

$$\boxed{R[w] = \{v \in W \mid wRv\}}$$

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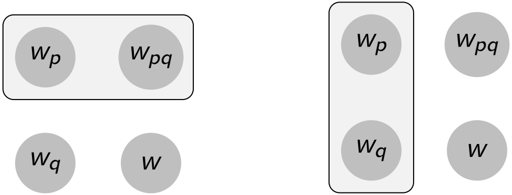
$$s \vDash p \iff \forall w \in s : w \in V(p)$$



(a) $s \vDash p$ (b) $s \nvDash p$

Syntax of *BSML*:

$$\phi ::= p \mid \neg\phi \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid \Diamond\phi \mid \mathrm{NE}$$

Semantics ($\vDash$)

$$\begin{array}{lcl}
s \vDash p & \iff & \forall w \in s : w \in V(p) \\
s \vDash \neg\phi & \iff & s \,\text{⫤}\, \phi \\
s \vDash \phi \wedge \psi & \iff & s \vDash \phi \text{ and } s \vDash \psi \\
s \vDash \phi \vee \psi & \iff & \exists t, t' : t \cup t' = s \text{ and } t \vDash \phi \text{ and } t' \vDash \psi \\
s \vDash \Diamond\phi & \iff & \forall w \in s : \exists t \subseteq R[w] : t \neq \varnothing \text{ and } t \vDash \phi \\
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Syntax of *BSML*:

$$\phi ::= p \mid \neg\phi \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid \Diamond \phi \mid \mathrm{NE}$$

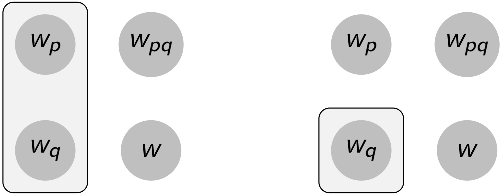
Semantics ($\models$)

$$\begin{aligned}
s \models p &\iff \forall w \in s : w \in V(p)\\
s \models \neg\phi &\iff s \dashv \phi\\
s \models \phi \wedge \psi &\iff s \models \phi \text{ and } s \models \psi\\
s \models \phi \vee \psi &\iff \exists t, t' : t \cup t' = s \text{ and } t \models \phi \text{ and } t' \models \psi\\
s \models \Diamond\phi &\iff \forall w \in s : \exists t \subseteq R[w] : t \neq \varnothing \text{ and } t \models \phi\\
s \models \mathrm{NE} &\iff s \neq \varnothing
\end{aligned}$$

$$\boxed{R[w] = \{v \in W \mid wRv\}}$$

Tensor disjunction $\vee$

$$s \vDash \phi \vee \psi \iff \exists t, t' : \begin{array}{ll} t \cup t' = s & \text{and} \\ t \vDash \phi & \text{and} \\ t' \vDash \psi & \end{array}$$



(a) $s \vDash p \vee q$ (b) $s \vDash p \vee q$

The non-emptiness atom NE

$$s \vDash \mathrm{NE} \iff s \neq \varnothing$$

(a) $s \vDash (p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE})$ (b) $s \nvDash (p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE})$

The modality $\Diamond$

$$\boxed{R[w] = \{v \in W \mid wRv\}}$$

$$s \vDash \Diamond\phi \quad \iff \quad \forall w \in s : \exists t \subseteq R[w] : t \neq \varnothing \text{ and } t \vDash \phi$$

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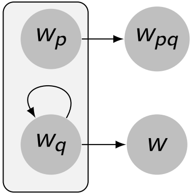
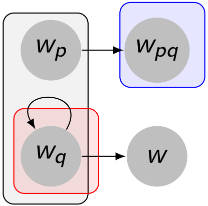


$s \vDash \Diamond q$ since

The modality $\Diamond$

$$\boxed{R[w] = \{v \in W \mid wRv\}}$$

$$s \vDash \Diamond\phi \quad \Longleftrightarrow \quad \forall w \in s : \exists t \subseteq R[w] : t \neq \varnothing \text{ and } t \vDash \phi$$



$s \vDash \Diamond q$ since

$\{w_q\} \subseteq R[w_q]$
$\{w_q\} \vDash q$

and

$\{w_{pq}\} \subseteq R[w_p]$
$\{w_{pq}\} \vDash q$

Syntax of *BSML*:

$$\phi ::= p \mid \neg\phi \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid \Diamond \phi \mid \mathrm{NE}$$

Semantics ($\models$)

$$\begin{array}{lcl}
s \models p & \Longleftrightarrow & \forall w \in s : w \in V(p) \\
s \models \neg\phi & \Longleftrightarrow & s \mathrel{=\!\!\!|} \phi \\
s \models \phi \wedge \psi & \Longleftrightarrow & s \models \phi \text{ and } s \models \psi \\
s \models \phi \vee \psi & \Longleftrightarrow & \exists t, t' : t \cup t' = s \text{ and } t \models \phi \text{ and } t' \models \psi \\
s \models \Diamond\phi & \Longleftrightarrow & \forall w \in s : \exists t \subseteq R[w] : t \neq \varnothing \text{ and } t \models \phi \\
s \models \mathrm{NE} & \Longleftrightarrow & s \neq \varnothing
\end{array}$$

$$\boxed{R[w] = \{v \in W \mid wRv\}}$$

Accounting for FC

The empty team $\varnothing$ supports contradictions such as $p \wedge \neg p$

NE represents lack of contradiction

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FC is caused by an intrusion of the pragmatic principle "avoid stating a contradiction" (NE) into meaning composition:

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FC is caused by an intrusion of the pragmatic principle "avoid stating a contradiction" (NE) into meaning composition:

$$\begin{aligned}
p^+ &:= p \wedge \mathrm{NE}\\
(\neg\phi)^+ &:= \neg\phi^+ \wedge \mathrm{NE}\\
(\phi\wedge\psi)^+ &:= (\phi^+ \wedge \psi^+) \wedge \mathrm{NE}\\
(\phi\vee\psi)^+ &:= (\phi^+ \vee \psi^+) \wedge \mathrm{NE}\\
(\Diamond\phi)^+ &:= \Diamond\phi^+ \wedge \mathrm{NE}
\end{aligned}$$

You may have coffee or tea.

$\rightsquigarrow$You may have coffee and you may have tea.

$$(\Diamond(c \vee t))^{+} \vDash \Diamond c \wedge \Diamond t$$

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$$(\Diamond(c \vee t))^{+} \vDash \Diamond c \wedge \Diamond t$$

i.e. $\Diamond(((c \wedge \mathrm{NE}) \vee (t \wedge \mathrm{NE})) \wedge \mathrm{NE}) \wedge \mathrm{NE} \vDash \Diamond c \wedge \Diamond t$

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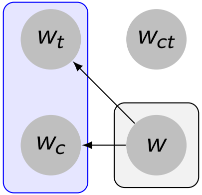
$$(\Diamond(c \vee t))^{+} \vDash \Diamond c \wedge \Diamond t$$

$$\text{i.e. } \Diamond(((c \wedge \mathrm{NE}) \vee (t \wedge \mathrm{NE})) \wedge \mathrm{NE}) \wedge \mathrm{NE} \vDash \Diamond c \wedge \Diamond t$$

$$\text{i.e. } \Diamond((c \wedge \mathrm{NE}) \vee (t \wedge \mathrm{NE})) \vDash \Diamond c \wedge \Diamond t$$

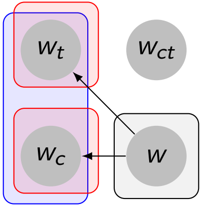
$$\Diamond\left((c \wedge \mathrm{NE}) \vee (t \wedge \mathrm{NE})\right) \qquad \models \qquad \Diamond c \wedge \Diamond t$$

$$\Diamond((c \wedge \mathrm{NE}) \vee (t \wedge \mathrm{NE})) \qquad \vDash \qquad \Diamond c \wedge \Diamond t$$



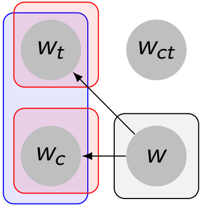
$\{w\} \vDash \Diamond((c \wedge \mathrm{NE}) \vee (t \wedge \mathrm{NE}))$ since

$$\Diamond((c \land \mathrm{NE}) \lor (t \land \mathrm{NE})) \qquad \vDash \qquad \Diamond c \land \Diamond t$$



$\{w\} \vDash \Diamond((c \land \mathrm{NE}) \lor (t \land \mathrm{NE}))$ since $\{w_c\} \vDash c$ and $\{w_t\} \vDash t$

$$\Diamond((c \wedge \mathrm{NE}) \vee (t \wedge \mathrm{NE})) \qquad \vDash \qquad \Diamond c \wedge \Diamond t$$



$\{w\} \vDash \Diamond((c \wedge \mathrm{NE}) \vee (t \wedge \mathrm{NE}))$ since $\{w_c\} \vDash c$ and $\{w_t\} \vDash t$

for the same reason, $\{w\} \vDash \Diamond c \wedge \Diamond t$

$BSML^{\mathbb{W}}$: $BSML$ with the global disjunction $\mathbb{W}$

$$s \models \phi \mathbin{\mathbb{W}} \psi \quad \iff \quad s \models \phi \text{ or } s \models \psi$$

BSML$^{\vvee}$: *BSML* with the global disjunction $\vvee$

$$s \models \phi \vvee \psi \quad\iff\quad s \models \phi \text{ or } s \models \psi$$

BSML$^{\oslash}$: *BSML* with the emptiness/circle operator $\oslash$

$$s \models \oslash\phi \quad\iff\quad s \models \phi \text{ or } s = \varnothing$$

$BSML^{\varovee}$: $BSML$ with the global disjunction $\varovee$

$$s \models \phi \varovee \psi \quad \Longleftrightarrow \quad s \models \phi \text{ or } s \models \psi$$

$BSML^{\oslash}$: $BSML$ with the emptiness/circle operator $\oslash$

$$s \models \oslash\phi \quad \Longleftrightarrow \quad s \models \phi \text{ or } s = \varnothing$$

For classical formulas α (no $\mathrm{NE}, \varovee, \oslash$):

$$s \models \alpha \iff \forall w \in s : \{w\} \models \alpha \iff \forall w \in s : w \models \alpha$$

Semantics ($\Dashv$)

$$\begin{array}{lcl}
s \Dashv p & \iff & \forall w \in s : w \notin V(p) \\
s \Dashv \neg\phi & \iff & s \vDash \phi \\
s \Dashv \phi \wedge \psi & \iff & \exists t, t' : t \cup t' = s \text{ and } t \Dashv \phi \text{ and } t' \Dashv \psi \\
s \Dashv \phi \vee \psi & \iff & s \Dashv \phi \text{ and } s \Dashv \psi \\
s \Dashv \phi \mathbin{\mathbb{W}} \psi & \iff & s \Dashv \phi \text{ and } s \Dashv \psi \\
s \Dashv \Diamond\phi & \iff & \forall w \in s : R[w] \Dashv \phi \\
s \Dashv \mathrm{NE} & \iff & s = \varnothing \\
s \Dashv \oslash\phi & \iff & s \Dashv \phi
\end{array}$$

Semantics ($\dashv$)

$$\begin{array}{lcl}
s \dashv p & \iff & \forall w \in s : w \notin V(p) \\
s \dashv \neg\phi & \iff & s \models \phi \\
s \dashv \phi \wedge \psi & \iff & \exists t, t' : t \cup t' = s \text{ and } t \dashv \phi \text{ and } t' \dashv \psi \\
s \dashv \phi \vee \psi & \iff & s \dashv \phi \text{ and } s \dashv \psi \\
s \dashv \phi \mathbin{\text{⩖}} \psi & \iff & s \dashv \phi \text{ and } s \dashv \psi \\
s \dashv \Diamond\phi & \iff & \forall w \in s : R[w] \dashv \phi \\
s \dashv \mathrm{NE} & \iff & s = \varnothing \\
s \dashv \oslash\phi & \iff & s \dashv \phi
\end{array}$$

$\Box := \neg \Diamond \neg$

$$s \models \Box\phi \iff \forall w \in s : R[w] \models \phi$$

Semantics ($\Dashv$)

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s \Dashv p & \iff & \forall w \in s : w \notin V(p) \\
s \Dashv \neg\phi & \iff & s \models \phi \\
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s \Dashv \mathrm{NE} & \iff & s = \varnothing \\
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\end{array}$$

$\neg\alpha$ behaves classically when α is classical

$$\Box := \neg \Diamond \neg$$

$$s \models \Box\phi \iff \forall w \in s : R[w] \models \phi$$

$$\begin{array}{ll}
\neg\neg\phi \equiv \phi & \neg(\phi \vee \psi) \equiv \neg\phi \wedge \neg\psi \\
\neg\mathrm{NE} \equiv p \wedge \neg p & \neg(\phi \wedge \psi) \equiv \neg\phi \vee \neg\psi \\
\neg \oslash \phi \equiv \neg\phi & \neg(\phi \Vee \psi) \equiv \neg\phi \wedge \neg\psi \\
\neg \Diamond \phi \equiv \Box\neg\phi &
\end{array}$$

You may not have coffee or tea.

$\rightsquigarrow$ You may not have coffee and you may not have tea.

$$\neg\Diamond(c \vee t) \rightarrow (\neg\Diamond c \wedge \neg\Diamond t)$$

You may not have coffee or tea.

$\leadsto$ You may not have coffee and you may not have tea.

$$\neg\Diamond(c \vee t) \rightarrow (\neg\Diamond c \wedge \neg\Diamond t)$$

In *BSML*: $(\neg\Diamond(b \vee c))^{+} \vDash \neg\Diamond b \wedge \neg\Diamond c$.

For classical α: $\alpha \equiv \oslash(\alpha \wedge \mathrm{NE})$

Using $\oslash$ we can define a function which cancels pragmatic enrichment:

$$
\begin{array}{lll}
p^- & := & \oslash p \\
\mathrm{NE}^- & := & \oslash \mathrm{NE} \\
(\neg\phi)^- & := & \oslash \neg\phi^- \\
(\phi \wedge \psi)^- & := & \oslash \phi^- \wedge \oslash\psi^- \\
(\phi \vee \psi)^- & := & \oslash \phi^- \vee \oslash\psi^- \\
(\Diamond\phi)^- & := & \oslash \Diamond\phi^-
\end{array}
$$

For classical $\alpha : (\alpha^+)^- \equiv \alpha$

Closure properties

ϕ is *downward closed*:	$[M, s \vDash \phi$ and $t \subseteq s] \implies M, t \vDash \phi$
ϕ is *union closed*:	$[M, s \vDash \phi$ for all $s \in S \neq \varnothing] \implies M, \bigcup S \vDash \phi$
ϕ has the *empty team property*:	$M, \varnothing \vDash \phi$ for all M
ϕ is *flat*:	$M, s \vDash \phi \iff M, \{w\} \vDash \phi$ for all $w \in s$

flat $\iff$ downward closed & union closed & empty team property

Classical formulas are flat

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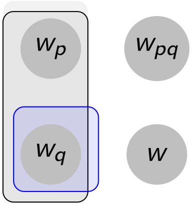
Formulas with NE may lack downward closure and the empty team property:



$$\{w_p, w_q\} \models (p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE})$$
$$\{w_q\} \not\models (p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE})$$

Classical formulas are flat

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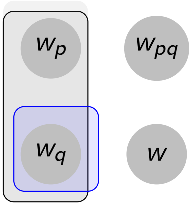
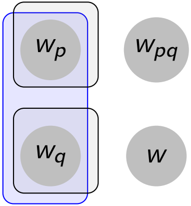


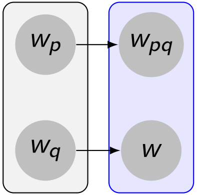
$$\begin{array}{lcl} \{w_p, w_q\} & \models & (p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE}) \\ \{w_q\} & \not\models & (p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE}) \end{array}$$

Formulas with $\mathbb{W}$ may lack union closure:



$$\begin{array}{lcl} \{w_p\} & \models & p \mathbin{\mathbb{W}} q \\ \{w_q\} & \models & p \mathbin{\mathbb{W}} q \\ \{w_p, w_q\} & \not\models & p \mathbin{\mathbb{W}} q \end{array}$$

The modal dependence logic modalities ◈ and ⧈

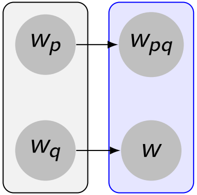


t is a successor team of s

$sRt :\Longleftrightarrow t \subseteq R[s]$ and $R[w] \cap t \neq \varnothing$ for all $w \in s$

$R[s] = \{v \in W \mid \exists w \in s : wRv\}$

The modal dependence logic modalities ◈ and ⧈



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$sRt :\Longleftrightarrow\ t \subseteq R[s]$ and $R[w] \cap t \neq \varnothing$ for all $w \in s$

$R[s] = \{v \in W \mid \exists w \in s : wRv\}$

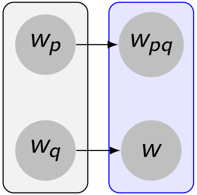
$$s \models \text{◈}\phi \quad\Longleftrightarrow\quad \exists t : sRt \text{ and } t \models \phi$$

$$s \models \text{⧈}\phi \quad\Longleftrightarrow\quad R[s] \models \phi$$

$$s \models \Diamond\phi \quad\Longleftrightarrow\quad \forall w \in s : \exists t \subseteq R[w] : t \models \phi \text{ and } t \neq \varnothing$$

$$s \models \Box\phi \quad\Longleftrightarrow\quad \forall w \in s : R[w] \models \phi$$

The modal dependence logic modalities $◈$ and $⧈$



t is a successor team of s

$sRt :\iff t \subseteq R[s]$ and $R[w] \cap t \neq \varnothing$ for all $w \in s$

$R[s] = \{v \in W \mid \exists w \in s : wRv\}$

$$s \models ◈\phi \iff \exists t : sRt \text{ and } t \models \phi$$

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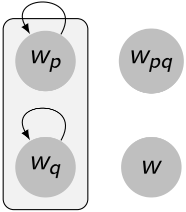
If ϕ is downward closed, $◈\phi \models \Diamond\phi$ and $⧈\phi \models \Box\phi$

If ϕ is union closed and has the empty team property, $\Diamond\phi \models ◈\phi$ and $\Box\phi \models ⧈\phi$

If ϕ is flat, $\Diamond\phi \equiv ◈\phi$ and $\Box\phi \equiv ⧈\phi$

Aloni's free choice explanation does not work with $\Diamond\!\!\!\Diamond$:



$$sRs \text{ and } s \models (p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE})$$
$$\text{Therefore } s \models \Diamond\!\!\!\Diamond((p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE}))$$

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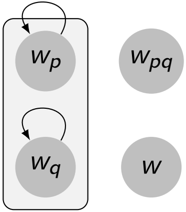

sRs and $s \vDash (p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE})$
Therefore $s \vDash \Diamond\!\!\!\!\Diamond((p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE}))$

s is the the only successor team of s and $s \nvDash p$
Therefore $s \nvDash \Diamond\!\!\!\!\Diamond p$ so $\Diamond\!\!\!\!\Diamond((p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE})) \nvDash \Diamond\!\!\!\!\Diamond p \wedge \Diamond\!\!\!\!\Diamond q$

Expressive Power

Fix a finite set of proposition symbols Φ

Pointed team model: (M, s) where M is a model over Φ; s is a team on M

Team property: set of pointed team models

$||\phi|| := \{(M, s) \mid M, s \vDash \phi\}$

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Theorem

$$\{||\phi|| \mid \phi \in BSML^{\mathbb{W}}\}$$

$$=$$

$$\{\text{property } P \mid P \text{ is invariant under team } k\text{-bisimulation for some } k \in \mathbb{N}\}$$

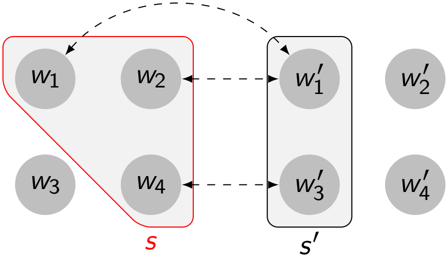
Team bisimulation:

$$s \rightleftharpoons_k s' :\Longleftrightarrow$$

Team bisimulation:

$s \leftrightarroweq_k s' :\Longleftrightarrow$

forth: $\forall w \in s : \exists w' \in s' : w \leftrightarroweq_k w'$



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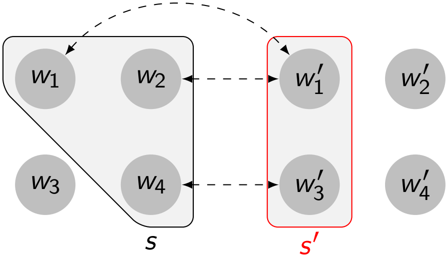

Team bisimulation:

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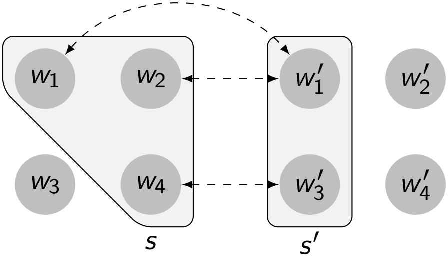
$s \leftrightarrows s' :\Longleftrightarrow \forall k \in \mathbb{N} : s \leftrightarrows_k s'$



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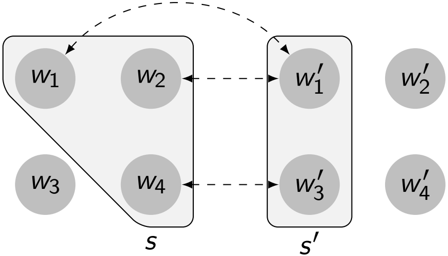

Theorem (bisimulation invariance)

$$s \leftrightarroweq_k s' \quad \Longrightarrow \quad s \equiv^k s'$$
$$s \leftrightarroweq s' \quad \Longrightarrow \quad s \equiv s'$$

Property P is *invariant under team k-bisimulation*:

$$[(M,s) \in P \text{ and } M,s \leftrightarroweq_k M',s'] \implies (M',s') \in P$$

Theorem

$$\{\|\phi\| \mid \phi \in BSML^{\mathbb{W}}\}$$
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$$\{\text{property } P \mid P \text{ is invariant under team } k\text{-bisimulation for some } k \in \mathbb{N}\}$$

Characteristic formulas for worlds
(Hintikka formulas):

$$\chi^{0}_{M,w} \quad := \quad \bigwedge\{p \mid w \in V(p)\} \wedge \bigwedge\{\neg p \mid w \notin V(p)\} \quad (p \in \Phi)$$

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$$w' \vDash \chi^k_w \iff w \leftrightarrows_k w'$$

Characteristic formulas for teams:

$$\begin{aligned}
\Theta^k_{M,s} &:= \bot & &\text{if } s = \varnothing \quad (\bot := p \wedge \neg p)\\
\Theta^k_{M,s} &:= \bigvee_{w \in s} (\chi^k_{M,w} \wedge \mathrm{NE}) & &\text{if } s \neq \varnothing
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Characteristic formulas for worlds (Hintikka formulas):

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Characteristic formulas for teams:

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$$s' \vDash \Theta^{k}_{s} \iff s \leftrightarroweq_{k} s'$$

$$\Theta_s^k = \bigvee_{w \in s} (\chi_w^k \wedge \mathrm{NE}) \qquad s' \models \Theta_s^k \iff s \leftrightarrows_k s'$$

$$\Theta_s^k = \bigvee_{w \in s} (\chi_w^k \wedge \mathrm{NE}) \qquad\qquad s' \models \Theta_s^k \iff s \rightleftarrows_k s'$$

Proof.

Case 1 : $s = \varnothing$.
Then $\Theta_s^k = \bot$ and
$s \rightleftarrows_k s' \iff s' = \varnothing \iff s' \models \bot$.

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So $\forall w \in s : \exists \{w'\} \subseteq s' : \{w'\} \vDash \chi_w^k \wedge \mathrm{NE}$.
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Therefore $s' \vDash \bigvee_{w \in s} (\chi_w^k \wedge \mathrm{NE})$, i.e. $s' \vDash \Theta_s^k$.

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$$\Theta_s^k = \bigvee_{w \in s} (\chi_w^k \wedge \mathrm{NE}) \qquad\qquad s' \vDash \Theta_s^k \iff s \leftrightarroows_k s'$$

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Let $w \in s$. Let $w' \in s'$ be s.t. $w \mathrel{\underline{\leftrightarrow}}_k w'$.
Then $w' \vDash \chi_w^k$
so $\{w'\} \vDash \chi_w^k$ and $\{w'\} \vDash \chi_w^k \land \mathrm{NE}$.

So $\forall w \in s : \exists \{w'\} \subseteq s' : \{w'\} \vDash \chi_w^k \land \mathrm{NE}$.
And $\forall \{w'\} \subseteq s' : \exists w \in s : \{w'\} \vDash \chi_w^k \land \mathrm{NE}$.
Therefore $s' \vDash \bigvee_{w \in s} (\chi_w^k \land \mathrm{NE})$, i.e. $s' \vDash \Theta_s^k$.

$\Longrightarrow$:
$s' \vDash \Theta_s^k$ implies there are $s'_w \subseteq s'$ $(\forall w \in s)$ s.t.
$s'_w \vDash \chi_w^k \land \mathrm{NE}$ and $s' = \bigcup_{w \in s} s'_w$.

Want to show:
$\forall w' \in s' : \exists w \in s : w \mathrel{\underline{\leftrightarrow}}_k w'$
$\forall w \in s : \exists w' \in s' : w \mathrel{\underline{\leftrightarrow}}_k w'$.

Let $w' \in s'$. Then for some $w \in s$ we have $w' \in s'_w$. Since $s'_w \vDash \chi_w^k \land \mathrm{NE}$, we have $s'_w \vDash \chi_w^k$. So $w' \vDash \chi_w^k$ whence $w \mathrel{\underline{\leftrightarrow}} w'$.

Let $w \in s$. Then there is some $s'_w \subseteq s'$ such that $s'_w \vDash \chi_w^k \land \mathrm{NE}$.

$$\Theta_s^k = \bigvee_{w \in s} (\chi_w^k \wedge \mathrm{NE}) \qquad\qquad s' \vDash \Theta_s^k \iff s \rightleftarrows_k s'$$

Proof.

Case 1 : $s = \varnothing$.
Then $\Theta_s^k = \bot$ and
$s \rightleftarrows_k s' \iff s' = \varnothing \iff s' \vDash \bot$.

Case 2: $s \neq \varnothing$.

$\Longleftarrow$:
Let $w \in s$. Let $w' \in s'$ be s.t. $w \rightleftarrows_k w'$.
Then $w' \vDash \chi_w^k$
so $\{w'\} \vDash \chi_w^k$ and $\{w'\} \vDash \chi_w^k \wedge \mathrm{NE}$.

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And $\forall \{w'\} \subseteq s' : \exists w \in s : \{w'\} \vDash \chi_w^k \wedge \mathrm{NE}$.
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$$\Theta_s^k = \bigvee_{w \in s} (\chi_w^k \wedge \mathrm{NE}) \qquad\qquad s' \models \Theta_s^k \iff s \mathrel{\underline{\leftrightarrow}}_k s'$$

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Let $w \in s$. Let $w' \in s'$ be s.t. $w \mathrel{\underline{\leftrightarrow}}_k w'$.
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And $\forall \{w'\} \subseteq s' : \exists w \in s : \{w'\} \models \chi_w^k \wedge \mathrm{NE}$.
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☐

Characteristic formulas for properties
(disjunctive normal form):

for P invariant under k-bisimulation:

$$M', s' \vDash \bigvee_{(M,s) \in P} \Theta_s^k \iff (M', s') \in P$$

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Theorem

$$\{\|\phi\| \mid \phi \in BSML^{\mathbb{W}}\}$$
$$=$$
$$\{\text{property } P \mid P \text{ is invariant under team } k\text{-bisimulation for some } k \in \mathbb{N}\}$$

Property P is *union closed*:

$$\{(M, s_i) \mid i \in I \neq \emptyset\} \subseteq P \implies (M, \bigcup_{i \in I} s_i) \subseteq P$$

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$$\mathbb{U} := \{P \mid P \text{ is union closed and invariant under } k\text{-bisimulation for some } k \in \mathbb{N}\}$$

$BSML$ is union closed, but not expressively complete for
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Lemma

For $\phi \in BSML$: ϕ has the empty team property $\implies \phi$ is downward closed.

BSML is union closed, but not expressively complete for
$\mathbb{U} = \{P \mid P$ is union closed and invariant under k-bisimulation for some $k \in \mathbb{N}\}$.

Lemma

For $\phi \in$ *BSML*: ϕ has the empty team property $\implies \phi$ is downward closed.

Consider $||(p \wedge \mathrm{NE}) \vee (\neg p \wedge \mathrm{NE})|| \cup ||\bot|| \in \mathbb{U}$.
Assume $||\psi|| = ||(p \wedge \mathrm{NE}) \vee (\neg p \wedge \mathrm{NE})||$ for $\psi \in$ *BSML*.

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If $\{w_p, w_{\neg p}\} \vDash (p \wedge \text{NE}) \vee (\neg p \wedge \text{NE})$, then $\{w_p, w_{\neg p}\} \vDash \psi$.

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If $\{w_p, w_{\neg p}\} \vDash (p \wedge \mathrm{NE}) \vee (\neg p \wedge \mathrm{NE})$, then $\{w_p, w_{\neg p}\} \vDash \psi$.
By downward closure $\{w_p\} \vDash \psi$.
$\{w_p\} \in ||(p \wedge \mathrm{NE}) \vee (\neg p \wedge \mathrm{NE})|| \cup ||\bot||$, a contradiction.

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By downward closure $\{w_p\} \vDash \psi$.
$\{w_p\} \in ||(p \wedge \mathrm{NE}) \vee (\neg p \wedge \mathrm{NE})|| \cup ||\bot||$, a contradiction.

In $BSML^{\oslash}$: $||(p \wedge \mathrm{NE}) \vee (\neg p \wedge \mathrm{NE})|| \cup ||\bot|| = || \oslash ((p \wedge \mathrm{NE}) \vee (\neg p \wedge \mathrm{NE}))||$

$$s' \models \Theta^k_s \quad \Longleftrightarrow \quad s \leftrightarrows_k s'$$

$$s' \models \oslash\Theta^k_s \quad \Longleftrightarrow \quad s \leftrightarrows_k s' \text{ or } s = \varnothing$$

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Characteristic formulas for union-closed properties with the empty team property:

$$M', s' \vDash \bigvee_{(M,s)\in P} \oslash\Theta_s^k \Longleftrightarrow (M', s') \in P$$

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Theorem

$$\{||\phi|| \mid \phi \in BSML^{\oslash}\}$$

$$=$$

$$\{P \mid P \text{ is union closed and invariant under } k\text{-bisimulation for some } k \in \mathbb{N}\}$$

For P union-closed; with the empty team property; invariant under k-bisimulation:

$$M', s' \vDash \bigvee_{(M,s)\in P} \oslash\Theta^k_s \iff (M', s') \in P$$

Proof.

$\Longleftarrow$: Let $(M', s') \in P$.

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Proof.

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$s' \vDash \bigvee_{(M,s)\in P} \oslash\Theta^k_s$.

$\Longrightarrow$: Assume $s' \vDash \bigvee_{M,s\in P} \oslash\Theta^k_s$.

Then $s' = \bigcup T$ where
$\forall t \in T : t = \varnothing$ or $\exists (M_t, s_t) \in P : t \vDash \Theta^k_{s_t}$

For P union-closed; with the empty team property; invariant under k-bisimulation:

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If $\forall t : t = \varnothing$, then $s' = \varnothing \in P$ by the empty team property.

For P union-closed; with the empty team property; invariant under k-bisimulation:

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If $\forall t : t = \varnothing$, then $s' = \varnothing \in P$ by the empty team property.

Otherwise let $T' = \{t \in T \mid t \neq \varnothing\}$ and consider $M = \biguplus\{M_t \mid t \in T'\}$ and its team $u = \bigcup\{s_t \mid t \in T'\}$.

For P union-closed; with the empty team property; invariant under k-bisimulation:

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$s' \models \Theta_{s'}^k$, so $s' \models \oslash\Theta_{s'}^k$.

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$s' = s' \cup \varnothing$.

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If $\forall t : t = \varnothing$, then $s' = \varnothing \in P$ by the empty team property. □

Otherwise let $T' = \{t \in T \mid t \neq \varnothing\}$ and consider $M = \biguplus\{M_t \mid t \in T'\}$ and its team $u = \bigcup\{s_t \mid t \in T'\}$.
$(M, u) \in P$ by invariance and union closure.
And $s' \mathrel{\underline{\leftrightarrow}}_k u$ so $s' \in P$.

Theorem

$$\{||\phi|| \mid \phi \in \text{BSMLI}\}$$
$$=$$
$$\{P \mid P \text{ is invariant under } k\text{-bisimulation for some } k \in \mathbb{N}\}$$

Characteristic formulas: $\displaystyle \mathbb{W}_{(M,s)\in P} \Theta^k_s$

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$BSML^{\mathbb{W}}$ axiomatization

α and β: classical formulas (no NE or $\mathbb{W}$).

Non-modal portion (adapted from the system for PT^+):

$\neg$ introduction

$$\dfrac{\begin{matrix}[\alpha]\\ D^*\\ \bot\end{matrix}}{\neg\alpha}\,\neg I(*)$$

$\neg$ elimination

$$\dfrac{\begin{matrix}D_1\\ \alpha\end{matrix}\qquad\begin{matrix}D_2\\ \neg\alpha\end{matrix}}{\beta}\,\neg E$$

$(*)$ The undischarged assumptions in D^* do not contain NE.

$\wedge$ introduction

$$\frac{\begin{array}{c}D_1\\ \phi\end{array}\qquad\begin{array}{c}D_2\\ \psi\end{array}}{\phi\wedge\psi}\wedge I$$

$\wedge$ elimination

$$\frac{\begin{array}{c}D\\ \phi\wedge\psi\end{array}}{\phi}\wedge E\qquad\qquad\frac{\begin{array}{c}D\\ \phi\wedge\psi\end{array}}{\psi}\wedge E$$

$\Vvee$ introduction

$$\frac{\begin{array}{c}D\\ \phi\end{array}}{\phi\Vvee\psi}\Vvee I\qquad\qquad\frac{\begin{array}{c}D\\ \psi\end{array}}{\phi\Vvee\psi}\Vvee I$$

$\Vvee$ elimination

$$\frac{\begin{array}{c}D\\ \phi\Vvee\psi\end{array}\qquad\begin{array}{c}[\phi]\\ D_1\\ \chi\end{array}\qquad\begin{array}{c}[\psi]\\ D_2\\ \chi\end{array}}{\chi}\Vvee E$$

$\vee$ weak introduction

$$\dfrac{\begin{array}{c} D \\ \phi \end{array}}{\phi \vee \psi}\, \vee I(**)$$

$\vee$ weakening

$$\dfrac{\begin{array}{c} D \\ \phi \end{array}}{\phi \vee \phi}\, \vee W$$

$\vee$ weak elimination

$$\dfrac{\begin{array}{c} \\ D \\ \phi \vee \psi \end{array} \qquad \begin{array}{c} [\phi] \\ D_1^* \\ \chi \end{array} \qquad \begin{array}{c} [\psi] \\ D_2^* \\ \chi \end{array}}{\chi}\, \vee E(*, \dagger)$$

$\vee$ weak substitution

$$\dfrac{\begin{array}{c} \\ D \\ \phi \vee \psi \end{array} \qquad \begin{array}{c} [\psi] \\ D_1^* \\ \chi \end{array}}{\phi \vee \chi}\, \vee Sub(*)$$

$(*)$ The undischarged assumptions in D_1^*, D_2^* do not contain NE.
$(**)$ ψ may not contain NE.
$(\dagger)$ χ may not contain $\mathbb{W}$ outside the scope of a $\Diamond$.

$\vee$ commutativity

$$\begin{array}{c} D \\ \dfrac{\phi \vee \psi}{\psi \vee \phi}\ \mathit{Com}\vee \end{array}$$

$\vee$ associativity

$$\begin{array}{c} D \\ \dfrac{(\phi \vee \psi) \vee \chi}{\phi \vee (\psi \vee \chi)}\ \mathit{Ass}\vee \end{array}$$

$\vee\mathbin{\vee\!\!\vee}$ distributivity

$$\begin{array}{c} D \\ \dfrac{\phi \vee (\psi \mathbin{\vee\!\!\vee} \chi)}{(\phi \vee \psi) \mathbin{\vee\!\!\vee} (\phi \vee \chi)}\ \mathit{Distr} \vee \mathbin{\vee\!\!\vee} \end{array}$$

⊥ Elimination

$$\frac{\begin{matrix} D \\ \phi \vee \bot \end{matrix}}{\phi} \bot E$$

$\bot\!\!\!\bot := \bot \wedge \mathrm{NE}$

$\bot\!\!\!\bot$ elimination

$$\frac{\begin{matrix} D \\ \bot\!\!\!\bot \end{matrix}}{\phi} \bot\!\!\!\bot\ E$$

$\bot\!\!\!\bot$ contraction

$$\frac{\begin{matrix} D \\ \bot\!\!\!\bot \vee \phi \end{matrix}}{\psi} \bot\!\!\!\bot\ Ctr$$

$\textsc{ne}$ introduction

$$\frac{}{\bot \mathbin{\mathbb{W}} \textsc{ne}}\ \textsc{ne}I$$

$\lor\textsc{ne}$ elimination

$$\frac{\begin{matrix} \\ D \\ \phi \lor \psi \end{matrix} \quad \begin{matrix} [\phi] \\ D_1 \\ \chi \end{matrix} \quad \begin{matrix} [\psi] \\ D_2 \\ \chi \end{matrix} \quad \begin{matrix} [(\phi \land \textsc{ne}) \lor (\psi \land \textsc{ne})] \\ D_3 \\ \chi \end{matrix}}{\chi}\ \lor\textsc{ne}E$$

New rules for $\neg$:

¬NE elimination

$$\begin{array}{c} D \\ \neg\mathrm{NE} \\ \hline\hline \bot \end{array}\ \neg\mathrm{NE}E$$

Double $\neg$ elimination

$$\begin{array}{c} D \\ \neg\neg\phi \\ \hline\hline \phi \end{array}\ DN$$

De Morgan 1

$$\begin{array}{c} D \\ \neg(\phi \wedge \psi) \\ \hline\hline \neg\phi \vee \neg\psi \end{array}\ DM_1$$

De Morgan 2

$$\begin{array}{c} D \\ \neg(\phi \vee \psi) \\ \hline\hline \neg\phi \wedge \neg\psi \end{array}\ DM_2$$

De Morgan 3

$$\begin{array}{c} D \\ \neg(\phi \mathbin{\vee\mkern-10mu\vee} \psi) \\ \hline\hline \neg\phi \wedge \neg\psi \end{array}\ DM_3$$

Modal portion—basic rules:

$\Diamond$ monotonicity

$$\dfrac{\begin{array}{c}[\phi]\\ D'\\ \psi\end{array} \qquad \begin{array}{c}D\\ \Diamond\phi\end{array}}{\Diamond\psi}\,\Diamond Mon(*)$$

$\Box$ monotonicity

$$\dfrac{\begin{array}{c}[\phi_1]\dots[\phi_n]\\ D'\\ \psi\end{array} \qquad \begin{array}{c}D_1\\ \Box\phi_1\end{array} \qquad \dots \qquad \begin{array}{c}D_n\\ \Box\phi_n\end{array}}{\Box\psi}\,\Box Mon(*)$$

$\Diamond\Box$ interaction

$$\begin{array}{c}D\\ \neg\Diamond\phi\\ \hline\hline \Box\neg\phi\end{array}\; Inter\,\Diamond\,\Box$$

$(*)$ D' does not contain undischarged assumptions.

New modal rules:

$\Diamond \mathbin{\mathbb{W}} \vee$ conversion

$$\dfrac{\begin{matrix} D \\ \Diamond(\phi \mathbin{\mathbb{W}} \psi) \end{matrix}}{\Diamond\phi \vee \Diamond\psi}\ \mathit{Conv} \Diamond \mathbb{W}\vee$$

$\Box \mathbin{\mathbb{W}} \vee$ conversion

$$\dfrac{\begin{matrix} D \\ \Box(\phi \mathbin{\mathbb{W}} \psi) \end{matrix}}{\Box\phi \vee \Box\psi}\ \mathit{Conv} \Box \mathbb{W}\vee$$

$\Diamond$ separation

$$\frac{\begin{array}{c} D \\ \Diamond(\phi \vee (\psi \wedge \mathrm{NE})) \end{array}}{\Diamond\psi}\,\Diamond Sep$$

$\Diamond$ join

$$\frac{\begin{array}{cc} D_1 & D_2 \\ \Diamond\phi & \Diamond\psi \end{array}}{\Diamond(\phi \vee \psi)}\,\Diamond Join$$

$\Box$ instantiation

$$\frac{\begin{array}{c} D \\ \Box(\phi \wedge \mathrm{NE}) \end{array}}{\Diamond\phi}\,\Box Inst$$

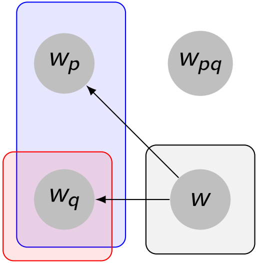
$\Box\Diamond$ join

$$\frac{\begin{array}{cc} D_1 & D_2 \\ \Box\phi & \Diamond\psi \end{array}}{\Box(\phi \vee \psi)}\,\Box\Diamond Join$$

$s \models \Diamond\phi \iff \forall w \in s : \exists t \subseteq R[w] : t \neq \varnothing \text{ and } t \models \phi$

$s \models \Box\phi \iff \forall w \in s : R[w] \models \phi$

$$\cfrac{\overset{D}{\Diamond(\phi \vee (\psi \wedge \mathrm{NE}))}}{\Diamond\psi}\ \Diamond Sep$$



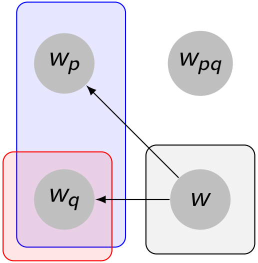
$s \vDash \Diamond(p \vee (q \wedge \mathrm{NE}))$

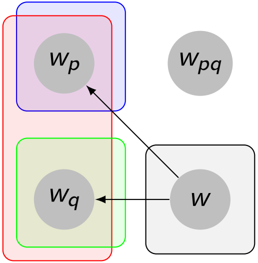
$s \vDash \Diamond q$

$s \vDash \Diamond\phi \iff \forall w \in s : \exists t \subseteq R[w] : t \neq \varnothing$ and $t \vDash \phi$

$s \vDash \Box\phi \iff \forall w \in s : R[w] \vDash \phi$

$$\dfrac{\overset{D}{\Diamond(\phi \vee (\psi \wedge \mathrm{NE}))}}{\Diamond\psi}\,\Diamond\mathit{Sep} \qquad\qquad \dfrac{\overset{D_1}{\Diamond\phi} \qquad \overset{D_2}{\Diamond\psi}}{\Diamond(\phi \vee \psi)}\,\Diamond\mathit{Join}$$





$s \models \Diamond(p \vee (q \wedge \mathrm{NE}))$

$s \models \Diamond q$

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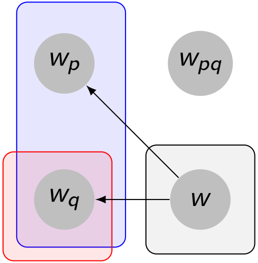

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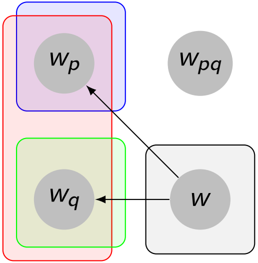

$s \vDash \Diamond p \wedge \Diamond q$

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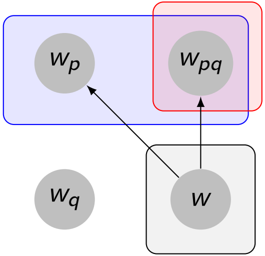

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$$\Diamond((\phi \wedge \mathrm{NE}) \vee (\psi \wedge \mathrm{NE})) \quad \dashv\vdash \quad \Diamond\phi \wedge \Diamond\psi \quad \mathit{FC}$$

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D

$$\frac{\Box(\phi \wedge \mathrm{NE})}{\Diamond\phi}\,\Box \mathit{Inst}$$

Corresponds to $(\Box\phi)^{+} \vDash \Diamond\phi$
—“ought implies may” for pragmatically enriched formulas.

Completeness

Use the disjunctive normal form and a strategy developed in [7]:

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Lemma: $\boxed{\phi \in BSML^{\mathbb{W}} \quad \Longrightarrow \quad \forall k \geq \text{modal depth}(\phi) : \exists P : \quad \phi \dashv\vdash \bigvee_{(M,s) \in P} \Theta^k_s}$

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$$\phi \vDash \psi \implies \bigvee_{(M,s) \in P} \Theta_s^k \vDash \bigvee_{(N,t) \in Q} \Theta_t^k$$

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$$\Longrightarrow \bigvee_{(M,s) \in P} \Theta_s^k \vdash \bigvee_{(N,t) \in Q} \Theta_t^k \quad \Longrightarrow \quad \phi \vdash \psi$$

$BSML^{\oslash}$ axiomatization

Exclude $\mathbb{W}$-rules and $\vee\mathrm{NE}E$; and add:

$BSML^{\oslash}$ axiomatization

Exclude ⩖-rules and $\vee\mathrm{NE}E$; and add:

$$\oslash\phi \equiv \phi ⩖ \bot$$

$BSML^{\oslash}$

$\oslash$ introduction

$$\cfrac{\begin{matrix} D \\ \bot \end{matrix}}{\oslash\phi}\ \oslash I \qquad \cfrac{\begin{matrix} D \\ \phi \end{matrix}}{\oslash\phi}\ \oslash I$$

$BSML^{⩖}$

⩖ introduction

$$\cfrac{\begin{matrix} D \\ \phi \end{matrix}}{\phi ⩖ \psi}\ ⩖ I \qquad \cfrac{\begin{matrix} D \\ \psi \end{matrix}}{\phi ⩖ \psi}\ ⩖ I$$

$BSML^{\oslash}$ axiomatization

Exclude $\Vee$-rules and $\vee\mathrm{NE}E$; and add:

$$\oslash\phi \equiv \phi \Vee \bot$$

$BSML^{\oslash}$

$\oslash$ introduction

$$\begin{array}{c} D \\ \dfrac{\bot}{\oslash\phi}\,\oslash I \end{array} \qquad \begin{array}{c} D \\ \dfrac{\phi}{\oslash\phi}\,\oslash I \end{array}$$

$\oslash\mathrm{NE}$ introduction

$$\dfrac{}{\oslash\mathrm{NE}}\,\oslash\mathrm{NE}I$$

$BSML^{\Vee}$

$\Vee$ introduction

$$\begin{array}{c} D \\ \dfrac{\phi}{\phi \Vee \psi}\,\Vee I \end{array} \qquad \begin{array}{c} D \\ \dfrac{\psi}{\phi \Vee \psi}\,\Vee I \end{array}$$

NE introduction

$$\dfrac{}{\bot \Vee \mathrm{NE}}\,\mathrm{NE}I$$

$BSML^{\oslash}$ axiomatization

Exclude ⩖-rules and $\vee\mathrm{NE}E$; and add:

$$\oslash\phi \equiv \phi \mathbin{⩖} \bot \qquad\qquad \neg\oslash\phi \equiv \neg\phi$$

$BSML^{\oslash}$

$\oslash$ introduction

$$\begin{array}{c} D \\ \bot \\ \hline \oslash\phi \end{array}\ \oslash I \qquad\qquad \begin{array}{c} D \\ \phi \\ \hline \oslash\phi \end{array}\ \oslash I$$

$\oslash\mathrm{NE}$ introduction

$$\begin{array}{c} \\ \hline \oslash\mathrm{NE} \end{array}\ \oslash\mathrm{NE}I$$

$\neg\oslash$ introduction

$$\begin{array}{c} D \\ \neg\phi \\ \hline\hline \neg\oslash\phi \end{array}\ \neg\oslash I$$

$BSML^{⩖}$

⩖ introduction

$$\begin{array}{c} D \\ \phi \\ \hline \phi \mathbin{⩖} \psi \end{array}\ ⩖ I \qquad\qquad \begin{array}{c} D \\ \psi \\ \hline \phi \mathbin{⩖} \psi \end{array}\ ⩖ I$$

NE introduction

$$\begin{array}{c} \\ \hline \bot \mathbin{⩖} \mathrm{NE} \end{array}\ \mathrm{NE}I$$

$BSML^{\oslash}$

$\oslash$ elimination

$$\dfrac{\begin{array}{c}D\\ \phi\end{array} \qquad \begin{array}{c}[\phi(\bot/[\oslash\psi, m])]\\ D_1\\ \chi\end{array} \qquad \begin{array}{c}[\phi(\psi/[\oslash\psi, m])]\\ D_2\\ \chi\end{array}}{\chi}\ \oslash E(*)$$

$(*)$ The occurrence at index m is not within the scope of $\neg$ or $\Diamond$.

$BSML^{⩔}$

⩔ elimination

$$\dfrac{\begin{array}{c}D\\ \phi \mathbin{⩔} \psi\end{array} \qquad \begin{array}{c}[\phi]\\ D_1\\ \chi\end{array} \qquad \begin{array}{c}[\psi]\\ D_2\\ \chi\end{array}}{\chi}\ {⩔}E$$

$BSML^{\oslash}$

$\Diamond\oslash$ elimination

$$\dfrac{\begin{array}{c} D \\ \Diamond\phi \end{array} \qquad \begin{array}{c} [\phi(\bot/[\oslash\psi, m])] \\ D_1 \\ \chi_1 \end{array} \qquad \begin{array}{c} [\phi(\psi/[\oslash\psi, m])] \\ D_2 \\ \chi_2 \end{array}}{\Diamond\chi_1 \vee \Diamond\chi_2}\ \Diamond\oslash E(*)$$

$(*)$ The occurrence at index m is not within the scope of a modality which occurs in ϕ, and not within the scope of $\neg$ (except if the $\neg$ forms part of $\Box$).

D_1, D_2 do not contain undischarged assumptions.

$BSML^{\mathbb{W}}$

$\Diamond \mathbb{W} \vee$ conversion

$$\dfrac{\begin{array}{c} D \\ \Diamond(\phi \mathbb{W} \psi) \end{array}}{\Diamond\phi \vee \Diamond\psi}\ Conv\,\Diamond\,\mathbb{W}\vee$$

$BSML^{\oslash}$

$\Diamond\oslash$ elimination

$$\dfrac{\begin{array}{c}D\\ \Diamond\phi\end{array}\qquad \begin{array}{c}[\phi(\bot/[\oslash\psi,m])]\\ D_1\\ \chi_1\end{array}\qquad \begin{array}{c}[\phi(\psi/[\oslash\psi,m])]\\ D_2\\ \chi_2\end{array}}{\Diamond\chi_1\vee\Diamond\chi_2}\;\Diamond\oslash E(*)$$

$\Box\oslash$ elimination

$$\dfrac{\begin{array}{c}D\\ \Box\phi\end{array}\qquad \begin{array}{c}[\phi(\bot/[\oslash\psi,m])]\\ D_1\\ \chi_1\end{array}\qquad \begin{array}{c}[\phi(\psi/[\oslash\psi,m])]\\ D_2\\ \chi_2\end{array}}{\Box\chi_1\vee\Box\chi_2}\;\Box\oslash E(*)$$

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$\Box\ \mathbb{W}\ \vee$ conversion

$$\dfrac{\begin{array}{c}D\\ \Box(\phi\,\mathbb{W}\,\psi)\end{array}}{\Box\phi\vee\Box\psi}\;Conv\,\Box\,\mathbb{W}\vee$$

Completeness

Lemma:

$$\phi \in \text{BSMLE} \implies \forall k \geq \text{md}(\phi) : \exists P : \quad \phi \dashv\vdash \bigvee_{(M,s)\in P} \oslash\Theta_s^k \quad \text{or} \quad \phi \dashv\vdash (\bigvee_{(M,s)\in P} \oslash\Theta_s^k) \wedge \mathrm{NE}$$

Completeness

Lemma:

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$\phi \vDash \psi$

Completeness

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$$\implies \forall (M,s) \in P : \exists R \subseteq Q : \quad s \leftrightarrows_k \biguplus R$$

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$$\boxed{\bigvee_{(N,t)\in Q} \oslash\Theta^k_t \equiv \mathbb{W}_{R\subseteq Q} \Theta^k_{\biguplus R}}$$

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$$\phi \vDash \psi \quad \implies \quad \bigvee_{(M,s)\in P} \oslash\Theta^k_s \vDash \bigvee_{(N,t)\in Q} \oslash\Theta^k_t$$

$$\implies \forall (M,s) \in P : \exists R \subseteq Q : \quad \begin{array}{l} s \rightleftarrows_k \biguplus R \\ \Theta^k_s \vdash \bigvee_{(N,t)\in R} \oslash\Theta^k_t \\ \Theta^k_s \vdash \bigvee_{(N,t)\in Q} \oslash\Theta^k_t \\ \oslash\Theta^k_s \vdash \bigvee_{(N,t)\in Q} \oslash\Theta^k_t \end{array}$$

$$\boxed{\bigvee_{(N,t)\in Q} \oslash\Theta^k_t \equiv \mathbb{W}_{R\subseteq Q} \Theta^k_{\biguplus R}}$$

$$\implies \bigvee_{(M,s)\in P} \oslash\Theta^k_s \vdash \bigvee_{(N,t)\in Q} \oslash\Theta^k_t$$

Completeness

Lemma:

$$\boxed{\phi \in \text{BSMLE} \implies \forall k \geq \text{md}(\phi) : \exists P : \quad \phi \dashv\vdash \bigvee_{(M,s)\in P} \oslash\Theta^k_s \quad \text{or} \quad \phi \dashv\vdash \Big(\bigvee_{(M,s)\in P} \oslash\Theta^k_s\Big) \wedge \mathrm{NE}}$$

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$$\implies \forall (M,s) \in P : \exists R \subseteq Q : \quad \begin{array}{l} s \rightleftharpoons_k \biguplus R \\ \Theta^k_s \vdash \displaystyle\bigvee_{(N,t)\in R} \oslash\Theta^k_t \\ \Theta^k_s \vdash \displaystyle\bigvee_{(N,t)\in Q} \oslash\Theta^k_t \\ \oslash\Theta^k_s \vdash \displaystyle\bigvee_{(N,t)\in Q} \oslash\Theta^k_t \end{array}$$

$$\boxed{\bigvee_{(N,t)\in Q} \oslash\Theta^k_t \equiv \mathop{\mathbb{W}}_{R\subseteq Q} \Theta^k_{\biguplus R}}$$

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BSML axiomatization

Exclude $\mathbb{W}$-rules and $\vee\text{NE}E$ from $BSML^{\mathbb{W}}$ and add:

BSML axiomatization

Exclude $\mathbb{W}$-rules and $\vee\mathrm{NE}E$ from $BSML^{\mathbb{W}}$ and add:

BSML

$\bot\mathrm{NE}$ translation

$$\dfrac{\begin{matrix} D \\ \phi \end{matrix} \qquad \begin{matrix} [\phi(\psi \wedge \bot/[\psi, m])] \\ D_1 \\ \chi \end{matrix} \qquad \begin{matrix} [\phi(\psi \wedge \mathrm{NE}/[\psi, m])] \\ D_2 \\ \chi \end{matrix}}{\chi}\ \bot\mathrm{NE}\,Trs(*)$$

$(*)$ The occurrence at index m is not within the scope of $\neg$ or $\Diamond$.

$BSML^{\mathbb{W}}$

NE
introduction

$$\dfrac{}{\bot \mathbb{W} \mathrm{NE}}\ \mathrm{NE}I$$

$BSML^{\emptyset}$

$\Diamond\bot\mathrm{NE}$ translation

$$\dfrac{\begin{array}{c} D \\ \Diamond\phi \end{array} \qquad \begin{array}{c}[\phi(\psi\wedge\bot/[\psi,m])]\\ D_1 \\ \chi_1\end{array} \qquad \begin{array}{c}[\phi(\psi\wedge \mathrm{NE}/[\psi,m])] \\ D_2 \\ \chi_2\end{array}}{\Diamond\chi_1 \vee \Diamond\chi_2}\ \Diamond\bot\mathrm{NE}\,\mathit{Trs}(*)$$

$(*)$ The occurrence at index m is not within the scope of a modality which occurs in ϕ, and not within the scope of $\neg$ (except if the $\neg$ forms part of $\Box$).

D_1, D_2 do not contain undischarged assumptions.

$BSML^{\mathbb{W}}$

$\Diamond \mathbb{W} \vee$ conversion

$$\dfrac{\begin{array}{c} D \\ \Diamond(\phi \mathbb{W} \psi) \end{array}}{\Diamond\phi \vee \Diamond\psi}\ \mathit{Conv}\,\Diamond\mathbb{W}\vee$$

$BSML^{\oslash}$

$\Diamond\bot\mathrm{NE}$ translation

$$\dfrac{\begin{array}{ccc} & [\phi(\psi\land\bot/[\psi,m])] & [\phi(\psi\land\mathrm{NE}/[\psi,m])] \\ D & D_1 & D_2 \\ \Diamond\phi & \chi_1 & \chi_2 \end{array}}{\Diamond\chi_1\lor\Diamond\chi_2}\;\Diamond\bot\mathrm{NE}\,Trs(*)$$

$\Box\bot\mathrm{NE}$ translation

$$\dfrac{\begin{array}{ccc} & [\phi(\psi\land\bot/[\psi,m])] & [\phi(\psi\land\mathrm{NE}/[\psi,m])] \\ D & D_1 & D_2 \\ \Box\phi & \chi_1 & \chi_2 \end{array}}{\Box\chi_1\lor\Box\chi_2}\;\Box\bot\mathrm{NE}\,Trs(*)$$

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$\Box\ \mathbb{W}\ \lor$ conversion

$$\dfrac{\begin{array}{c} D \\ \Box(\phi\mathbin{\mathbb{W}}\psi) \end{array}}{\Box\phi\lor\Box\psi}\;Conv\,\Box\,\mathbb{W}\lor$$

Old rules in [7]/[2] which are derivable:

$$\Gamma^k: \quad \text{set of all non-equivalent } \Theta^k_{s_i} \text{ over } \Phi, \text{ where } s_i \neq \varnothing$$
$$\mathrm{NE} \equiv \mathbb{W}\,\Gamma^k$$

BSML

NE elimination

$$\dfrac{\begin{array}{c} D \\ \phi \end{array} \qquad \begin{array}{c} [\phi(\Theta^k_{s_1}/[\mathrm{NE}, m])] \\ D_1 \\ \chi \end{array} \qquad \cdots \qquad \begin{array}{c} [\phi(\Theta^k_{s_n}/[\mathrm{NE}, m])] \\ D_n \\ \chi \end{array}}{\chi}\ \mathrm{NE}E(*)$$

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$BSML^{\mathbb{W}}$

$\mathbb{W}$ elimination

$$\dfrac{\begin{array}{c} D \\ \phi \mathbin{\mathbb{W}} \psi \end{array} \qquad \begin{array}{c} [\phi] \\ D_1 \\ \chi \end{array} \qquad \begin{array}{c} [\psi] \\ D_2 \\ \chi \end{array}}{\chi}\ \mathbb{W}E$$

$BSML$

$\Diamond\mathrm{NE}$ elimination

$$\dfrac{\begin{array}{cccc} & [\phi(\Theta^k_{s_1}/[\mathrm{NE},m])] & & [\phi(\Theta^k_{s_n}/[\mathrm{NE},m])] \\ D & D_1 & & D_n \\ \Diamond\phi & \chi_1 & \dots & \chi_n \end{array}}{\bigvee_{i\in I}\Diamond\chi_i}\,\Diamond\mathrm{NE}E(*)$$

$\Box\mathrm{NE}$ elimination

$$\dfrac{\begin{array}{cccc} & [\phi(\Theta^k_{s_1}/[\mathrm{NE},m])] & & [\phi(\Theta^k_{s_n}/[\mathrm{NE},m])] \\ D & D_1 & & D_n \\ \Box\phi & \chi_1 & \dots & \chi_n \end{array}}{\bigvee_{i\in I}\Box\chi_i}\,\Box\mathrm{NE}E(*)$$

$(*)$ The occurrence at index m is not within the scope of a modality which occurs in ϕ, and not within the scope of $\neg$ (except if the $\neg$ forms part of $\Box$); $D_1, \dots, D_n$ do not contain undischarged assumptions; $k \in \mathbb{N}$; $\{\Theta_{s_1}, \dots, \Theta_{s_n}\} = \Gamma^k$.

$BSML^{\mathbb{W}}$

$\Diamond \mathbin{\mathbb{W}} \vee$ conversion

$$\dfrac{\begin{array}{c} D \\ \Diamond(\phi \mathbin{\mathbb{W}} \psi) \end{array}}{\Diamond\phi \vee \Diamond\psi}\,Conv \Diamond \mathbin{\mathbb{W}}\vee$$

$\Box \mathbin{\mathbb{W}} \vee$ conversion

$$\dfrac{\begin{array}{c} D \\ \Box(\phi \mathbin{\mathbb{W}} \psi) \end{array}}{\Box\phi \vee \Box\psi}\,Conv \Box \mathbin{\mathbb{W}}\vee$$

$$\dfrac{\begin{array}{c} D \\ \phi \end{array} \qquad \begin{array}{c} [\phi(\Theta^k_{s_1}/[\mathrm{NE}, m])] \\ D_1 \\ \chi \end{array} \qquad \dots \qquad \begin{array}{c} [\phi(\Theta^k_{s_n}/[\mathrm{NE}, m])] \\ D_n \\ \chi \end{array}}{\chi}\ \mathrm{NE}E(*)$$

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Let $\chi^k_s := \bigvee_{w \in s} \chi^k_w$. By classical completeness $\vdash \chi^k_s$, where s is such that for each k–th Hintikka formula $\chi^k_{w'}$, there is a $w \in s$ such that $\chi^k_w \dashv\vdash \chi^k_{w'}$.

$$\dfrac{\begin{array}{c} D \\ \phi \end{array} \qquad \begin{array}{c} [\phi(\Theta^k_{s_1}/[\mathrm{NE},m])] \\ D_1 \\ \chi \end{array} \qquad \dots \qquad \begin{array}{c} [\phi(\Theta^k_{s_n}/[\mathrm{NE},m])] \\ D_n \\ \chi \end{array}}{\chi}\ \mathrm{NE}E(*)$$

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Then it can be shown that $\phi \vdash \phi(\chi^k_s \wedge \mathrm{NE}/[\mathrm{NE}, m])$. Let $\psi := \phi(\chi^k_s \wedge \mathrm{NE}/[\mathrm{NE}, m])$.

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Consider the case in which $|s| = 2$. Let $\chi^k_s = \chi^k_{w_1} \vee \chi^k_{w_2}$.

$$\dfrac{\begin{array}{c} D \\ \phi \end{array} \qquad \begin{array}{c} [\phi(\Theta^k_{s_1}/[\mathrm{NE}, m])] \\ D_1 \\ \chi \end{array} \qquad \cdots \qquad \begin{array}{c} [\phi(\Theta^k_{s_n}/[\mathrm{NE}, m])] \\ D_n \\ \chi \end{array}}{\chi}\,\mathrm{NE}E(*)$$

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Assume $\psi(\chi^k_{w_1} \wedge \bot/\chi^k_{w_1})(\chi^k_{w_2} \wedge \bot/\chi^k_{w_2})$ for $\bot\mathrm{NE}\,\mathit{Trs}$. This is equivalent to $\phi(\bot \wedge \mathrm{NE}/[\mathrm{NE}, m])$, which gives χ via the $\perp\!\!\!\perp$-rules.

$$\dfrac{\begin{array}{c} D \\ \phi \end{array} \qquad \begin{array}{c}[\phi(\Theta^k_{s_1}/[\mathrm{NE}, m])] \\ D_1 \\ \chi\end{array} \qquad \dots \qquad \begin{array}{c}[\phi(\Theta^k_{s_n}/[\mathrm{NE}, m])] \\ D_n \\ \chi\end{array}}{\chi}\ \mathrm{NE}E(*)$$

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$$\dfrac{\begin{matrix} & [\phi(\Theta^k_{s_1}/[\mathrm{NE},m])] & & [\phi(\Theta^k_{s_n}/[\mathrm{NE},m])] \\ D & D_1 & & D_n \\ \phi & \chi & \dots & \chi \end{matrix}}{\chi}\ \mathrm{NE}E(*)$$

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Similarly $\psi(\chi^k_{w_1} \wedge \mathrm{NE}/\chi^k_{w_1})(\chi^k_{w_2} \wedge \bot/\chi^k_{w_2}) \vdash \chi$ and $\psi(\chi^k_{w_1} \wedge \mathrm{NE}/\chi^k_{w_1})(\chi^k_{w_2} \wedge \mathrm{NE}/\chi^k_{w_2}) \vdash \chi$.

So $\psi \vdash \chi$ by iterated applications of $\bot\mathrm{NE}\,Trs$.

Completeness

The idea: simulate the disjunctive normal forms using "instantiations" [7]. In an instantiation ϕ_f for a formula ϕ, each atom η is replaced by some Θ^0_s such that $s \vDash \psi$:

$$\begin{array}{ccc} \phi & \Longrightarrow & \phi_f \\ p \vee (q \wedge \mathrm{NE}) & \Longrightarrow & \Theta^0_{s_p} \vee (\Theta^0_{s_q} \wedge \Theta^0_{\mathrm{NE}}) \end{array}$$

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F_ϕ: the set of all instantiations of ϕ

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F_ϕ: the set of all instantiations of ϕ

Since for each atom η we have $\psi \equiv \mathbb{W}_{(M,s)\in P} \Theta^0_s$, where $P = \|\eta\| = \{(M,s) \mid M,s \vDash \psi\}$, then assuming that $\mathbb{W}$ distributes over everything:

$$\phi \equiv \mathbb{W} F_\phi$$

And given rules that simulate $\mathbb{W}$:

$$\forall \phi_f \in F_\phi : \phi_f \vdash \phi$$
$$\text{if } \forall \phi_f \in F_\phi : \Gamma, \phi_f \vdash \psi, \text{ then } \Gamma, \phi \vdash \psi$$

Problem: in *BSML*, $\mathbin{\mathbb{W}}$ does not distribute over $\Diamond$. For instance $\Diamond(p \mathbin{\mathbb{W}} q) \not\models \Diamond p \mathbin{\mathbb{W}} \Diamond q$.

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Solution: we treat maximal modal subformulas as atoms: $p \wedge \Diamond \Box q \Longrightarrow \Theta^0_{s_p} \wedge \Theta^2_{s_{\Diamond \Box q}}$.

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Lemma ($\mathbb{W}$-distributive form): $\phi \in$ BSML implies $\phi \dashv\vdash \phi'$ where
ϕ' does not contain NE within the scope of a $\Diamond$, and is in negation normal form

Problem: in *BSML*, $\vee\!\!\!\vee$ does not distribute over $\Diamond$. For instance $\Diamond(p \vee\!\!\!\vee q) \not\models \Diamond p \vee\!\!\!\vee \Diamond q$.

Solution: we treat maximal modal subformulas as atoms: $p \wedge \Diamond \Box q \Longrightarrow \Theta^0_{s_p} \wedge \Theta^2_{s_{\Diamond \Box q}}$.

Lemma ($\vee\!\!\!\vee$-distributive form): $\phi \in \mathrm{BSML}$ implies $\phi \dashv\vdash \phi'$ where
ϕ' does not contain NE within the scope of a $\Diamond$, and is in negation normal form

An instantiation ϕ_f of ϕ in $\vee\!\!\!\vee$-distributive form:

each NE is replaced by some $\Theta^0_{s_f}$ where $s_f \models \mathrm{NE}$

each $\eta \in \{p, \neg p, \Diamond\psi, \Box\psi\}$ is replaced by some $\chi^k_{s_f} = \bigvee_{w \in s_f} \chi^k_w$ $(s_f \models \eta;\ k = md(\eta))$

Problem: in *BSML*, $\Vee$ does not distribute over $\Diamond$. For instance $\Diamond(p \Vee q) \not\models \Diamond p \Vee \Diamond q$.

Solution: we treat maximal modal subformulas as atoms: $p \wedge \Diamond \Box q \Longrightarrow \Theta^0_{s_p} \wedge \Theta^2_{s_{\Diamond\Box q}}$.

Lemma ($\Vee$-distributive form): $\phi \in$ BSML implies $\phi \dashv\vdash \phi'$ where
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An instantiation ϕ_f of ϕ in $\Vee$-distributive form:

- each NE is replaced by some $\Theta^0_{s_f}$ where $s_f \models \mathrm{NE}$
- each $\eta \in \{p, \neg p, \Diamond\psi, \Box\psi\}$ is replaced by some $\chi^k_{s_f} = \bigvee_{w \in s_f} \chi^k_w$ $(s_f \models \eta;\ k = md(\eta))$

$$\phi \equiv \bigVee F_\phi$$
$$\forall \phi_f \in F_\phi : \phi_f \vdash \phi$$
$$\text{if } \forall \phi_f \in F_\phi : \Gamma, \phi_f \vdash \psi, \text{ then } \Gamma, \phi \vdash \psi$$

$$\begin{aligned} & \phi \models \psi \\ \Longrightarrow \quad & \mathbb{W} F_\phi \models \mathbb{W} F_\psi \end{aligned}$$

$$\phi \models \psi$$

$$\Longrightarrow \quad \mathbb{W} F_\phi \models \mathbb{W} F_\psi$$

$$\boxed{\begin{array}{l} \forall \phi_f : \forall k \geq md(\phi) : \exists sf_1, sf_2 : \\ \phi_f \dashv\vdash \Theta^k_{sf_1} \vee \chi^k_{sf_2} \end{array}}$$

$$\phi \vDash \psi$$

$$\Longrightarrow \quad \mathbb{W} F_\phi \vDash \mathbb{W} F_\psi$$

$$\Longrightarrow \quad \mathop{\mathbb{W}}_{(sf_1,sf_2)\in A} \Theta^k_{sf_1} \vee \chi^k_{sf_2} \vDash \mathop{\mathbb{W}}_{(sg_1,sg_2)\in B} \Theta^k_{sg_1} \vee \chi^k_{sg_2}$$

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$$\phi \models \psi$$

$$\Longrightarrow \quad \mathbb{W} F_{\phi} \models \mathbb{W} F_{\psi}$$

$$\Longrightarrow \quad \mathop{\mathbb{W}}_{(sf_1,sf_2)\in A} \Theta^{k}_{sf_1} \vee \chi^{k}_{sf_2} \models \mathop{\mathbb{W}}_{(sg_1,sg_2)\in B} \Theta^{k}_{sg_1} \vee \chi^{k}_{sg_2}$$

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$$\Theta^{k}_{s_1} \vee \chi^{k}_{s_2} \equiv \mathop{\mathbb{W}}_{t\subseteq s_2} \Theta^{k}_{s_1 \uplus t}$$
$$\forall t \subseteq s_2 : \Theta^{k}_{s_1 \uplus t} \vdash \Theta^{k}_{s_1} \vee \chi^{k}_{s_2}$$
$$[\forall t \subseteq s_2 : \Gamma, \Theta^{k}_{s_1 \uplus t} \vdash \psi]$$
$$\Longrightarrow \Gamma, \Theta^{k}_{s_1} \vee \chi^{k}_{s_2} \vdash \psi$$

$$
\begin{aligned}
& \phi \models \psi \\
\Longrightarrow\quad & \mathbb{W} F_\phi \models \mathbb{W} F_\psi \\
\Longrightarrow\quad & \mathbb{W}_{(sf_1, sf_2) \in A} \Theta^k_{sf_1} \vee \chi^k_{sf_2} \models \mathbb{W}_{(sg_1, sg_2) \in B} \Theta^k_{sg_1} \vee \chi^k_{sg_2} \\
\Longrightarrow\quad & \mathbb{W}_{(sf_1, sf_2) \in A} \mathbb{W}_{t \subseteq sf_2} \Theta^k_{sf_1 \uplus t} \models \mathbb{W}_{(sg_1, sg_2) \in B} \mathbb{W}_{u \subseteq sg_2} \Theta^k_{sg_1 \uplus u}
\end{aligned}
$$

$\forall \phi_f : \forall k \geq md(\phi) : \exists sf_1, sf_2 :$

$\phi_f \dashv\vdash \Theta^k_{sf_1} \vee \chi^k_{sf_2}$

$$\Theta^k_{s_1} \vee \chi^k_{s_2} \equiv \mathbb{W}_{t \subseteq s_2} \Theta^k_{s_1 \uplus t}$$

$\forall t \subseteq s_2 : \Theta^k_{s_1 \uplus t} \vdash \Theta^k_{s_1} \vee \chi^k_{s_2}$

$[\forall t \subseteq s_2 : \Gamma, \Theta^k_{s_1 \uplus t} \vdash \psi]$

$\Longrightarrow \Gamma, \Theta^k_{s_1} \vee \chi^k_{s_2} \vdash \psi$

$$\phi \models \psi$$

$$\implies \quad \mathbb{W} F_\phi \models \mathbb{W} F_\psi$$

$$\implies \quad \mathbb{W}_{(sf_1,sf_2)\in A} \Theta^k_{sf_1} \vee \chi^k_{sf_2} \models \mathbb{W}_{(sg_1,sg_2)\in B} \Theta^k_{sg_1} \vee \chi^k_{sg_2}$$

$$\implies \quad \mathbb{W}_{(sf_1,sf_2)\in A} \mathbb{W}_{t\subseteq sf_2} \Theta^k_{sf_1 \uplus t} \models \mathbb{W}_{(sg_1,sg_2)\in B} \mathbb{W}_{u\subseteq sg_2} \Theta^k_{sg_1 \uplus u}$$

$$\implies \quad \forall (sf_1, sf_2) \in A, t \subseteq sf_2 : \exists (sg_1, sg_2) \in B, u \subseteq sg_2 : \Theta^k_{sf_1 \uplus t} \mathrel{\underline{\leftrightarrow}}_k \Theta^k_{sg_1 \uplus u}$$

$$\forall \phi_f : \forall k \geq md(\phi) : \exists sf_1, sf_2 : \phi_f \dashv\vdash \Theta^k_{sf_1} \vee \chi^k_{sf_2}$$

$$\Theta^k_{s_1} \vee \chi^k_{s_2} \equiv \mathbb{W}_{t\subseteq s_2} \Theta^k_{s_1 \uplus t}$$

$$\forall t \subseteq s_2 : \Theta^k_{s_1 \uplus t} \vdash \Theta^k_{s_1} \vee \chi^k_{s_2}$$

$$[\forall t \subseteq s_2 : \Gamma, \Theta^k_{s_1 \uplus t} \vdash \psi]$$

$$\implies \Gamma, \Theta^k_{s_1} \vee \chi^k_{s_2} \vdash \psi$$

$$\phi \models \psi$$

$$\Longrightarrow \quad \mathbb{W} F_\phi \models \mathbb{W} F_\psi$$

$$\Longrightarrow \quad \mathbb{W}_{(sf_1,sf_2)\in A} \Theta^k_{sf_1} \vee \chi^k_{sf_2} \models \mathbb{W}_{(sg_1,sg_2)\in B} \Theta^k_{sg_1} \vee \chi^k_{sg_2}$$

$$\Longrightarrow \quad \mathbb{W}_{(sf_1,sf_2)\in A} \mathbb{W}_{t\subseteq sf_2} \Theta^k_{sf_1 \uplus t} \models \mathbb{W}_{(sg_1,sg_2)\in B} \mathbb{W}_{u\subseteq sg_2} \Theta^k_{sg_1 \uplus u}$$

$$\Longrightarrow \quad \forall (sf_1, sf_2) \in A, t \subseteq sf_2 : \exists (sg_1, sg_2) \in B, u \subseteq sg_2 :$$

$$\Theta^k_{sf_1 \uplus t} \Leftrightarrow_k \Theta^k_{sg_1 \uplus u}$$

$$\Theta^k_{sf_1 \uplus t} \dashv\vdash \Theta^k_{sg_1 \uplus u} \vdash \Theta^k_{sg_1} \vee \chi^k_{sg_2} \vdash \psi_g \vdash \psi$$

$$\forall \phi_f : \forall k \geq md(\phi) : \exists sf_1, sf_2 :$$

$$\phi_f \dashv\vdash \Theta^k_{sf_1} \vee \chi^k_{sf_2}$$

$$\Theta^k_{s_1} \vee \chi^k_{s_2} \equiv \mathbb{W}_{t\subseteq s_2} \Theta^k_{s_1 \uplus t}$$

$$\forall t \subseteq s_2 : \Theta^k_{s_1 \uplus t} \vdash \Theta^k_{s_1} \vee \chi^k_{s_2}$$

$$[\forall t \subseteq s_2 : \Gamma, \Theta^k_{s_1 \uplus t} \vdash \psi]$$

$$\Longrightarrow \quad \Gamma, \Theta^k_{s_1} \vee \chi^k_{s_2} \vdash \psi$$

$$\phi \vDash \psi$$

$$\Longrightarrow \quad \mathbb{W} F_\phi \vDash \mathbb{W} F_\psi$$

$$\Longrightarrow \quad \mathop{\mathbb{W}}\limits_{(sf_1,sf_2)\in A} \Theta^k_{sf_1} \vee \chi^k_{sf_2} \vDash \mathop{\mathbb{W}}\limits_{(sg_1,sg_2)\in B} \Theta^k_{sg_1} \vee \chi^k_{sg_2}$$

$$\Longrightarrow \quad \mathop{\mathbb{W}}\limits_{(sf_1,sf_2)\in A} \mathop{\mathbb{W}}\limits_{t\subseteq sf_2} \Theta^k_{sf_1\uplus t} \vDash \mathop{\mathbb{W}}\limits_{(sg_1,sg_2)\in B} \mathop{\mathbb{W}}\limits_{u\subseteq sg_2} \Theta^k_{sg_1\uplus u}$$

$$\Longrightarrow \quad \forall (sf_1, sf_2) \in A, t \subseteq sf_2 : \exists (sg_1, sg_2) \in B, u \subseteq sg_2 :$$

$$\Theta^k_{sf_1\uplus t} \mathrel{\underline{\leftrightarrow}}_k \Theta^k_{sg_1\uplus u}$$

$$\Theta^k_{sf_1\uplus t} \dashv\vdash \Theta^k_{sg_1\uplus u} \vdash \Theta^k_{sg_1} \vee \chi^k_{sg_2} \vdash \psi_g \vdash \psi$$

$$\Longrightarrow \quad \forall (sf_1, sf_2) \in A : \Theta^k_{sf_1} \vee \chi^k_{sf_2} \vdash \psi$$

$$\forall \phi_f : \forall k \geq md(\phi) : \exists sf_1, sf_2 :$$

$$\phi_f \dashv\vdash \Theta^k_{sf_1} \vee \chi^k_{sf_2}$$

$$\Theta^k_{s_1} \vee \chi^k_{s_2} \equiv \mathop{\mathbb{W}}\limits_{t\subseteq s_2} \Theta^k_{s_1\uplus t}$$

$$\forall t \subseteq s_2 : \Theta^k_{s_1\uplus t} \vdash \Theta^k_{s_1} \vee \chi^k_{s_2}$$

$$[\forall t \subseteq s_2 : \Gamma, \Theta^k_{s_1\uplus t} \vdash \psi]$$

$$\Longrightarrow \quad \Gamma, \Theta^k_{s_1} \vee \chi^k_{s_2} \vdash \psi$$

$$\phi \vDash \psi$$

$$\Longrightarrow \quad \mathbb{W} F_\phi \vDash \mathbb{W} F_\psi$$

$$\Longrightarrow \quad \mathbb{W}_{(sf_1, sf_2) \in A} \Theta^k_{sf_1} \vee \chi^k_{sf_2} \vDash \mathbb{W}_{(sg_1, sg_2) \in B} \Theta^k_{sg_1} \vee \chi^k_{sg_2}$$

$$\Longrightarrow \quad \mathbb{W}_{(sf_1, sf_2) \in A} \mathbb{W}_{t \subseteq sf_2} \Theta^k_{sf_1 \uplus t} \vDash \mathbb{W}_{(sg_1, sg_2) \in B} \mathbb{W}_{u \subseteq sg_2} \Theta^k_{sg_1 \uplus u}$$

$$\Longrightarrow \quad \forall (sf_1, sf_2) \in A, t \subseteq sf_2 : \exists (sg_1, sg_2) \in B, u \subseteq sg_2 :$$
$$\Theta^k_{sf_1 \uplus t} \Leftrightarrow_k \Theta^k_{sg_1 \uplus u}$$
$$\Theta^k_{sf_1 \uplus t} \dashv\vdash \Theta^k_{sg_1 \uplus u} \vdash \Theta^k_{sg_1} \vee \chi^k_{sg_2} \vdash \psi_g \vdash \psi$$

$$\Longrightarrow \quad \forall (sf_1, sf_2) \in A : \Theta^k_{sf_1} \vee \chi^k_{sf_2} \vdash \psi$$

$$\Longrightarrow \quad \forall \phi_f \in F_\phi : \phi_f \vdash \psi$$

$$\forall \phi_f : \forall k \geq md(\phi) : \exists sf_1, sf_2 :$$
$$\phi_f \dashv\vdash \Theta^k_{sf_1} \vee \chi^k_{sf_2}$$

$$\Theta^k_{s_1} \vee \chi^k_{s_2} \equiv \mathbb{W}_{t \subseteq s_2} \Theta^k_{s_1 \uplus t}$$

$$\forall t \subseteq s_2 : \Theta^k_{s_1 \uplus t} \vdash \Theta^k_{s_1} \vee \chi^k_{s_2}$$

$$[\forall t \subseteq s_2 : \Gamma, \Theta^k_{s_1 \uplus t} \vdash \psi]$$
$$\Longrightarrow \quad \Gamma, \Theta^k_{s_1} \vee \chi^k_{s_2} \vdash \psi$$

$$
\begin{aligned}
& \phi \vDash \psi \\
\Longrightarrow\quad & \bigvee F_\phi \vDash \bigvee F_\psi \\
\Longrightarrow\quad & \bigvee_{(sf_1, sf_2) \in A} \Theta^k_{sf_1} \vee \chi^k_{sf_2} \vDash \bigvee_{(sg_1, sg_2) \in B} \Theta^k_{sg_1} \vee \chi^k_{sg_2} \\
\Longrightarrow\quad & \bigvee_{(sf_1, sf_2) \in A} \bigvee_{t \subseteq sf_2} \Theta^k_{sf_1 \uplus t} \vDash \bigvee_{(sg_1, sg_2) \in B} \bigvee_{u \subseteq sg_2} \Theta^k_{sg_1 \uplus u} \\
\Longrightarrow\quad & \forall (sf_1, sf_2) \in A, t \subseteq sf_2 : \exists (sg_1, sg_2) \in B, u \subseteq sg_2 : \\
& \Theta^k_{sf_1 \uplus t} \leftrightarrows_k \Theta^k_{sg_1 \uplus u} \\
& \Theta^k_{sf_1 \uplus t} \dashv\vdash \Theta^k_{sg_1 \uplus u} \vdash \Theta^k_{sg_1} \vee \chi^k_{sg_2} \vdash \psi_g \vdash \psi \\
\Longrightarrow\quad & \forall (sf_1, sf_2) \in A : \Theta^k_{sf_1} \vee \chi^k_{sf_2} \vdash \psi \\
\Longrightarrow\quad & \forall \phi_f \in F_\phi : \phi_f \vdash \psi \\
\Longrightarrow\quad & \phi \vdash \psi
\end{aligned}
$$

$$
\boxed{\begin{array}{l}
\forall \phi_f : \forall k \geq md(\phi) : \exists sf_1, sf_2 : \\
\phi_f \dashv\vdash \Theta^k_{sf_1} \vee \chi^k_{sf_2}
\end{array}}
$$

$$
\boxed{\begin{array}{l}
\Theta^k_{s_1} \vee \chi^k_{s_2} \equiv \bigvee_{t \subseteq s_2} \Theta^k_{s_1 \uplus t} \\
\forall t \subseteq s_2 : \Theta^k_{s_1 \uplus t} \vdash \Theta^k_{s_1} \vee \chi^k_{s_2} \\
[\forall t \subseteq s_2 : \Gamma, \Theta^k_{s_1 \uplus t} \vdash \psi] \\
\qquad\qquad \Longrightarrow \Gamma, \Theta^k_{s_1} \vee \chi^k_{s_2} \vdash \psi
\end{array}}
$$

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